

Quantum Graphs

Supervision:

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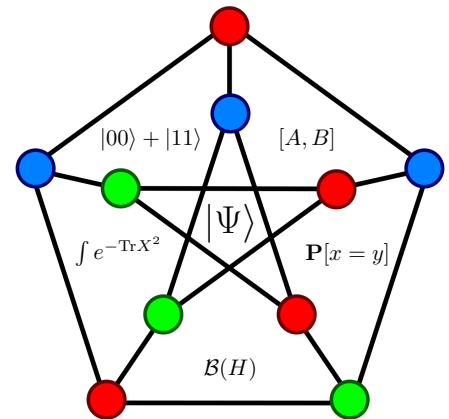
Duration: 3 years

Salary: French standard, approx. 1600 EUR

1 Background

The *Quantum Graphs* project proposes an in-depth exploration of the combinatorial aspects at the heart of quantum information theory, at the interface between combinatorics and theoretical computer science on the one hand, and quantum physics on the other. More specifically, our project aims at several theoretical developments of the notion of quantum graphs, seen as a non-commutative generalization of graphs. This interdisciplinary proposal aims to develop new combinatorial and algebraic methods for solving fundamental problems in quantum information, while shedding light on the deep relationships between combinatorial structures and quantum properties.

Within this framework of quantum information theory, the notion of quantum graphs (also called non-commutative graphs) was first introduced by Duan et al in [DSW13] with the aim of generalizing a certain notion in Shannon theory to the quantum case. As an analogue to the fact that classical graphs can be seen as non-reflexive symmetric relations, Weaver [Wea21] formulated quantum graphs as reflexive symmetric quantum relations over a von Neumann algebra. Musto et. al. [MRV18] also formulated finite quantum graphs as adjacency operators on finite quantum sets. Very surprisingly, these three different points of view lead to the same objects, *quantum graphs*, the focus of this PhD project.



2 Goals

Our project aims to define and study other *properties of quantum graphs*, such as connectivity (recently studied in [Mat23]), Ramsey theory, coloring problems, automorphism groups, and so on. These issues are fundamentally linked to central problems in quantum information theory. The initial approach of [DSW13] started from the observation that a quantum channel (a completely positive linear application that preserves the trace) naturally defines a quantum graph via its Kraus decomposition. Studying this graph, as in the classical case, makes it possible to calculate the channel's ability to transmit information

(whether classical or quantum) perfectly (without error, i.e. without the need for an encoding/decoding scheme). This was the first special case where the analogue of the classical properties of a quantum graph was of particular interest in computer science. Our first objective is to go beyond these first examples to draw up a more complete dictionary of combinatorial properties between classical and quantum graphs, always bearing in mind applications in theoretical physics, in Shannon's quantum theory, or in the theory of open quantum systems via the study of quantum channels.

In addition to a theoretical study of the properties of quantum graphs, the student will investigate the possibility of defining quantum versions of other well-known objects in computer science, such as *combinatorial maps* [Zvo97]. It should be recalled here that combinatorial maps are discrete objects that have been intensively studied both in theoretical computer science (at LaBRI but also elsewhere) and in theoretical physics, as they can be seen as Feynman graphs of random matrix models.

This way of proceeding has proved very fruitful in similar circumstances in combinatorics, for example in the case of quantum Latin squares [MV16] and in *SudoQ*, the quantum version of the Sudoku game [NP21].

An expected result of this project would therefore be to give a coherent definition of a quantum equivalent of combinatorial maps. Once this quantum generalization has been found, we shall turn our attention to the combinatorial properties of these objects, such as the number of faces, embeddings, etc. Within this new framework, our main objective will be to find quantum combinatorial maps that cannot be realized as classical maps and that have remarkable properties from the point of view of extremal combinatorics.

A final line of research we shall develop is motivated by *non-local games* in quantum information theory and the various notions of graph isomorphism and homomorphism (see the overview paper by Palazuelos and Vidick [PV16]). Non-local games are cooperative games in which two parties, Alice and Bob, must answer questions posed by a referee without communicating, trying to maximize a score. Depending on the resources at their disposal (classical, quantum, or super-quantum), Alice and Bob can win the game with greater or lesser probability. The violation of Bell's inequality, the experimental realization of which earned Alain Aspect the Nobel Prize in 2022, can be seen as the strict inequality between the classical maximum score and the quantum maximum score for a particular non-local game.

Recently, a non-local game was introduced by Atserias et. al. [AMR⁺19] to study the notion of (classical) graph isomorphism. Two graphs are isomorphic (in the usual sense of the term) if and only if Alice and Bob can win with certainty at the game associated with the graphs, using a classical strategy (they have at their disposal a common random variable). By giving them more power thanks to an entangled quantum state they share, Alice and Bob can win the same game for additional pairs of graphs, called quantum isomorphic graphs. Later, the same authors showed that two graphs are quantum isomorphic if and only if they have the same number of homomorphisms from all planar graphs [MR20].

Our final objective is to study the same questions for quantum graphs, by defining the notion of isomorphic quantum graphs. An expected result would be to find purely quantum examples in this framework, which do not correspond to classical graphs.

3 Candidate profile

We are looking for a PhD candidate who is highly motivated to work on problems at the interface between mathematics, physics and computer science. The candidate must have a master degree in either of these fields. They should be familiar with the mathematical formalism of quantum mechanics and have a solid knowledge of linear algebra and some notions of discrete mathematics and graph theory. Moreover, they should be willing to work with the variety of mathematical tools relevant to the project: multilinear algebra, combinatorial maps, operator algebra, graph theory, combinatorics, etc.

References

- [AMR⁺19] Albert Atserias, Laura Mančinská, David E Roberson, Robert Šámal, Simone Severini, and Antonios Varvitsiotis. Quantum and non-signalling graph isomorphisms. *Journal of Combinatorial Theory, Series B*, 136:289–328, 2019.
- [DSW13] Runyao Duan, Simone Severini, and Andreas Winter. Zero-error communication via quantum channels, noncommutative graphs, and a quantum lovász number. *IEEE Transactions on Information Theory*, 59(2):1164–1174, 2013.
- [Mat23] Junichiro Matsuda. Algebraic connectedness and bipartiteness of quantum graphs. *arXiv preprint arXiv:2310.09500*, 2023.
- [MR20] Laura Mančinská and David E Roberson. Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 661–672. IEEE, 2020.
- [MRV18] Benjamin Musto, David Reutter, and Dominic Verdon. A compositional approach to quantum functions. *Journal of Mathematical Physics*, 59(8):081706, 2018.
- [MV16] B Musto and J Vicary. Quantum latin squares and unitary error bases. *Quantum Information and Computation*, 16(15-16), 2016.
- [NP21] Ion Nechita and Jordi Pillet. Sudocq - a quantum variant of the popular game. *Quantum Information & Computation*, 21(9,10), 2021.
- [PV16] Carlos Palazuelos and Thomas Vidick. Survey on nonlocal games and operator space theory. *Journal of Mathematical Physics*, 57(1):015220, 2016.
- [Wea21] Nik Weaver. Quantum graphs as quantum relations. *The Journal of Geometric Analysis*, 31(9):9090–9112, 2021.
- [Zvo97] Alexander Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Mathematical and Computer Modelling*, 26(8-10):281–304, 1997.