

Computational aspects of the trace norm contraction coefficient

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Sending a *classical* bit through a channel

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Noise: Channel $\Phi : \mathcal{L}(\mathbb{C}^n) \rightarrow \mathcal{L}(\mathbb{C}^m)$.

Messages:

- Classical: $i \in [k] := \{1, \dots, k\}$ **Focus of this work**
- Quantum: $\rho \in \mathcal{L}(\mathbb{C}^k)$, $\rho \geq 0$, $\text{tr}(\rho) = 1$. (Work in progress...)

Reduces to: Compute $\mathbb{P}_{\text{succ}}(\Phi, k)$, the maximum probability of success for transmitting k messages.

Definition

$$\mathbb{P}_{\text{succ}}(\Phi, k) := \max_{\substack{\{M_i \mid 1 \leq i \leq k\} \text{ POVM,} \\ \{\rho_i \in \mathcal{D}(\mathcal{H}) \mid 1 \leq i \leq k\}}} \frac{1}{k} \sum_{i=1}^k \text{tr}(M_i \Phi(\rho_i)). \quad (1)$$

Special case: Classical channels.

Theorem ([BF17])

When Φ is classical:

- *Easy to approximate $\mathbb{P}_{\text{succ}}(\Phi, k)$ up to $(1 - e^{-1})$.*
- *NP-hard to do better.*

Encoding of a single bit

We focus on **encoding a bit**, i.e. $k = 2$.

- $\mathbb{P}_{\text{succ}}(\Phi, 2) = \frac{1}{2} + \frac{1}{2}\eta_{\text{tr}}(\Phi)$, with

$$\eta_{\text{tr}}(\Phi) := \max_{\substack{\rho, \sigma \in \mathcal{D}(\mathcal{H}) \\ \rho \neq \sigma}} \frac{\|\Phi(\rho) - \Phi(\sigma)\|_1}{\|\rho - \sigma\|_1}$$

- Quantifies the data-processing inequality
- Contraction coefficients well-studied for many information measures [Geo+25; HT24]...

Main results

Theorem 1

- It is NP-hard to approximate $\eta_{\text{tr}}(\Phi)$ to a factor $1/\sqrt{2} + \varepsilon$ for any $\varepsilon > 0$, even when Φ is unital.
- If Φ has classical output, the contraction coefficient is NP-hard to approximate to a factor $\sqrt{2/\pi} + \varepsilon$ for any $\varepsilon > 0$.

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Theorem 2

NP-complete to distinguish between $\eta_{\text{tr}}(\Phi) = 1$ and $\eta_{\text{tr}}(\Phi) \leq 1 - \Omega(\frac{1}{n^3})$.

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Theorem 2 NP-complete to distinguish between $\eta_{\text{tr}}(\Phi) = 1$ and $\eta_{\text{tr}}(\Phi) \leq 1 - \Omega(\frac{1}{n^3})$.

Theorem 3 There is a family SDP_m of size $n^{O(m)}$ such that

$$0 \leq \text{SDP}_m - \eta_{\text{tr}}(\Phi) \leq \frac{\text{poly}(n)}{\sqrt{m}}.$$

Proof Thm. 1 – Φ with Classical output

Theorem (Little Grothendieck Problem)

Input: A linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Objective: Compute $\|f\|_{2 \rightarrow 1} = \sup\{\|f(x)\|_1 \mid \|x\|_2 \leq 1\}$.

Hardness: NP-hard to approximate [BRS15; NRV13]

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Reduction sketch:

- For any channel Φ ,

$$\eta_{\text{tr}}(\Phi) = \frac{1}{2} \max_{X^* = X, \|X\|_\infty \leq 1} (\lambda_{\max} - \lambda_{\min})(\Phi^*(X)).$$
- Note that $\|f\|_{2 \rightarrow 1} = \sup_{\|y\|_\infty \leq 1} \|f^*(y)\|_2$.
- Choose $\Phi^*(y) = f^*(y)e^T + e f^*(y)^T$ with $e \perp f^*(y)$.
- Finally $\sup_{Y^* = Y, \|Y\|_\infty \leq 1} (\lambda_{\max} - \lambda_{\min})(\Phi^*(Y)) = 4\beta \sup_{\|y\|_\infty \leq 1} \|f^*(y)\|_2$.

Proof Thm. 2

Outlook: $\eta_{\text{tr}}(\Phi) = 1 \iff J(\Phi^* \circ \Phi)$ separable. Show that SEP is hard on instances of the form $J(\Phi^* \circ \Phi)$.

Core idea: Reduce Graph 2-CSP problem to the problem of deciding whether $J(\Phi^* \circ \Phi)$ is separable.

The $\text{QMA}_{\log}(2, a, b)$ complexity class

Definition $(\text{QMA}_{\log}(2, a, b))$

(L, \bar{L}) is in $\text{QMA}_{\log}(2, a, b)$ if there is a polynomial $p(m)$ and a polynomial-time classical verification algorithm \mathcal{V} that on input x of length m prepares a quantum circuit $\mathcal{V}(x)$ acting on $O(\log m)$ qubits such that for any m and any instance of size m , we have

- If $x \in L$, there are $u, v \in \mathbb{C}^{p(m)}$ unit vectors such that

$$\text{tr}(\Pi_{\mathcal{V}(x)} uu^* \otimes vv^*) \geq a$$

where $\Pi_{\mathcal{V}(x)}$ is the acceptance projector of $\mathcal{V}(x)$.

- If $x \in \bar{L}$, for any unit vectors $u, v \in \mathbb{C}^{p(m)}$, we have

$$\text{tr}(\Pi_{\mathcal{V}(x)} uu^* \otimes vv^*) \leq b.$$

The reduction

- Graph 2-CSP is in $\text{QMA}_{\text{log}}(2, 1, 1 - \frac{1}{n})$ [GNN12].
- For G instance of 2-CSP, take Π the acceptance projector of the quantum verifier [BT09; GNN12].
- G accepting equivalent to Π separable.
- Let $S = \text{vec}^{-1}(\ker(\Pi))$ and $\widehat{S} = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$.
- $\widehat{S}^\perp = G_\Phi$ the *quantum confusability graph* of Φ with

$$G_\Phi := \text{Span}\{K_i^* K_j \mid 1 \leq i, j \leq O(n^2)\}.$$

- $\eta_{\text{tr}}(\Phi) = 1 \Leftrightarrow \exists uv^* \in G_\Phi^\perp \Leftrightarrow G_\Phi$ has indep. number ≥ 2 [Dua09].
- Π separable $\Leftrightarrow \exists uv^* \in \widehat{S} = G_\Phi^\perp \Leftrightarrow \eta_{\text{tr}}(\Phi) = 1 \Leftrightarrow J(\Phi^* \circ \Phi)$ separable.

SDP hierarchy converging to $\mathbb{P}_{\text{succ}}(\Phi, k)$.

Theorem 3 proven using [Ber+21] for separability with linear constraints.

Remark:

- The size of our SDP hierarchy grows exponentially with the level.
- Quantum Doeblin Coefficient give efficient bounds (but not as tight) [Hir24; Geo+25]

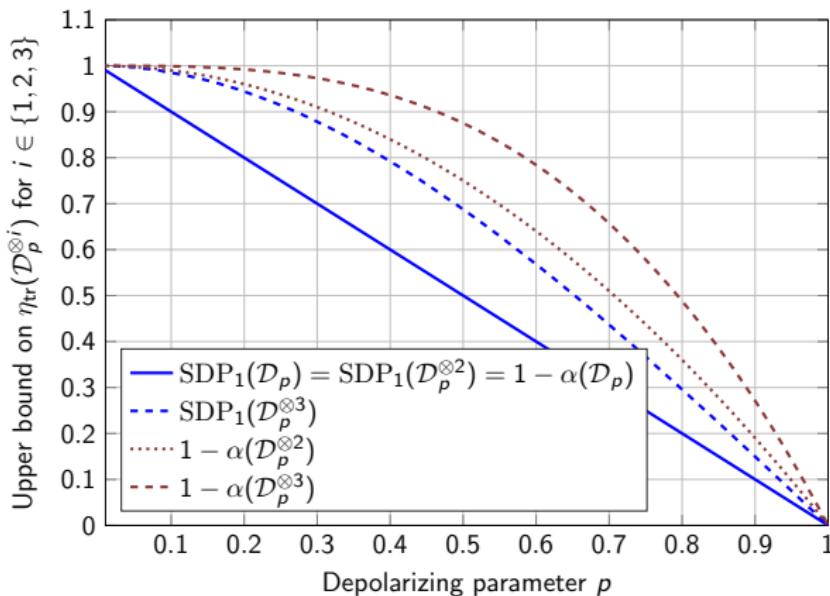


Figure: Upper bounds on the contraction coefficient of (multiple copies of) depolarizing channels obtained via the first level of the hierarchy we propose.

Conclusion

NP-hardness of:

- Approximate the trace norm contraction coefficient
- Find optimal encoding of a bit
- Decide if a quantum graph has independence number ≥ 2

Future work:

- Approximation algorithm based on separability results?
- Better algorithm if Φ is structured?
- Encoding of a qubit (work in progress...)

Thank you for your attention

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