

# Computational aspects of the trace norm contraction coefficient

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**Noise:** Channel  $\Phi : \mathcal{L}(\mathbb{C}^n) \rightarrow \mathcal{L}(\mathbb{C}^m)$ .

**Messages:**

- Classical:  $i \in [k] := \{1, \dots, k\}$  **Focus of this work**
- Quantum:  $\rho \in \mathcal{L}(C^k)$ ,  $\rho \geq 0$ ,  $\text{tr}(\rho) = 1$ . (Work in progress...)

**Reduces to:** Compute  $\mathbb{P}_{\text{succ}}(\Phi, k)$ , the maximum probability of success for transmitting  $k$  messages.

## Definition

$$\mathbb{P}_{\text{succ}}(\Phi, k) := \max_{\substack{\{M_i | 1 \leq i \leq k\} \text{ POVM,} \\ \{\rho_i \in \mathcal{D}(\mathcal{H}) | 1 \leq i \leq k\}}} \frac{1}{k} \sum_{i=1}^k \text{tr}(M_i \Phi(\rho_i)). \quad (1)$$



**Special case:** Classical channels.**Theorem ([BF17])**

*When  $\Phi$  is classical:*

- *Easy to approximate  $\mathbb{P}_{\text{succ}}(\Phi, k)$  up to  $(1 - e^{-1})$ .*
- *NP-hard to do better.*



# Encoding of a single bit

We focus on **encoding a bit**, i.e.  $k = 2$ .

- $\mathbb{P}_{\text{succ}}(\Phi, 2) = \frac{1}{2} + \frac{1}{2}\eta_{\text{tr}}(\Phi)$ , with

$$\eta_{\text{tr}}(\Phi) := \max_{\substack{\rho, \sigma \in \mathcal{D}(\mathcal{H}) \\ \rho \neq \sigma}} \frac{\|\Phi(\rho) - \Phi(\sigma)\|_1}{\|\rho - \sigma\|_1}$$

- Quantifies the data-processing inequality
- Contraction coefficients well-studied for many information measures [Geo+25; HT24]...



# Main results

## Theorem 1

- It is NP-hard to approximate  $\eta_{\text{tr}}(\Phi)$  to a factor  $1/\sqrt{2} + \varepsilon$  for any  $\varepsilon > 0$ , even when  $\Phi$  is unital.
- If  $\Phi$  has classical output, the contraction coefficient is NP-hard to approximate to a factor  $\sqrt{2/\pi} + \varepsilon$  for any  $\varepsilon > 0$ .



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**Theorem 2** NP-complete to distinguish between  $\eta_{\text{tr}}(\Phi) = 1$  and  $\eta_{\text{tr}}(\Phi) \leq 1 - \Omega(\frac{1}{n^3})$ .

**Theorem 3** There is a family  $\text{SDP}_m$  of size  $n^{O(m)}$  such that

$$0 \leq \text{SDP}_m - \eta_{\text{tr}}(\Phi) \leq \frac{\text{poly}(n)}{\sqrt{m}}.$$



# Proof Thm. 1 – $\Phi$ with Classical output

## Theorem (Little Grothendieck Problem)

*Input:* A linear function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

*Objective:* Compute  $\|f\|_{2 \rightarrow 1} = \sup\{\|f(x)\|_1 \mid \|x\|_2 \leq 1\}$ .

*Hardness:* NP-hard to approximate [BRS15; NRV13]



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## Reduction sketch:

- For any channel  $\Phi$ ,  

$$\eta_{\text{tr}}(\Phi) = \frac{1}{2} \max_{X^*=X, \|X\|_\infty \leq 1} (\lambda_{\max} - \lambda_{\min})(\Phi^*(X)).$$
- Note that  $\|f\|_{2 \rightarrow 1} = \sup_{\|y\|_\infty \leq 1} \|f^*(y)\|_2$ .
- Choose  $\Phi^*(y) = f^*(y)e^T + ef^*(y)^T$  with  $e \perp f^*(y)$ .
- Finally  $\sup_{Y^*=Y, \|Y\|_\infty \leq 1} (\lambda_{\max} - \lambda_{\min})(\Phi^*(Y)) = 4\beta \sup_{\|y\|_\infty \leq 1} \|f^*(y)\|_2$ .



## Proof Thm. 2

**Outlook:**  $\eta_{\text{tr}}(\Phi) = 1 \iff J(\Phi^* \circ \Phi)$  separable. Show that SEP is hard on instances of the form  $J(\Phi^* \circ \Phi)$ .

**Core idea:** Reduce Graph 2-CSP problem to the problem of deciding whether  $J(\Phi^* \circ \Phi)$  is separable.



# The $\text{QMA}_{\log}(2, a, b)$ complexity class

## Definition ( $\text{QMA}_{\log}(2, a, b)$ )

$(L, \bar{L})$  is in  $\text{QMA}_{\log}(2, a, b)$  if there is a polynomial  $p(m)$  and a polynomial-time classical verification algorithm  $\mathcal{V}$  that on input  $x$  of length  $m$  prepares a quantum circuit  $\mathcal{V}(x)$  acting on  $O(\log m)$  qubits such that for any  $m$  and any instance of size  $m$ , we have

- If  $x \in L$ , there are  $u, v \in \mathbb{C}^{p(m)}$  unit vectors such that

$$\text{tr}(\Pi_{\mathcal{V}(x)} uu^* \otimes vv^*) \geq a$$

where  $\Pi_{\mathcal{V}(x)}$  is the acceptance projector of  $\mathcal{V}(x)$ .

- If  $x \in \bar{L}$ , for any unit vectors  $u, v \in \mathbb{C}^{p(m)}$ , we have

$$\text{tr}(\Pi_{\mathcal{V}(x)} uu^* \otimes vv^*) \leq b.$$



# The reduction

- Graph 2-CSP is in  $\text{QMA}_{\log}(2, 1, 1 - \frac{1}{n})$  [GNN12].
- For  $G$  instance of 2-CSP, take  $\Pi$  the acceptance projector of the quantum verifier [BT09; GNN12].
- $G$  accepting equivalent to  $\Pi$  separable.
- Let  $S = \text{vec}^{-1}(\ker(\Pi))$  and  $\hat{S} = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$ .
- $\hat{S}^\perp = G_\Phi$  the *quantum confusability graph* of  $\Phi$  with

$$G_\Phi := \text{Span}\{K_i^* K_j \mid 1 \leq i, j \leq O(n^2)\}.$$

- $\eta_{\text{tr}}(\Phi) = 1 \Leftrightarrow \exists uv^* \in G_\Phi^\perp \Leftrightarrow G_\Phi$  has indep. number  $\geq 2$  [Dua09].
- $\Pi$  separable  $\Leftrightarrow \exists uv^* \in \hat{S} = G_\Phi^\perp \Leftrightarrow \eta_{\text{tr}}(\Phi) = 1 \Leftrightarrow J(\Phi^* \circ \Phi)$  separable.



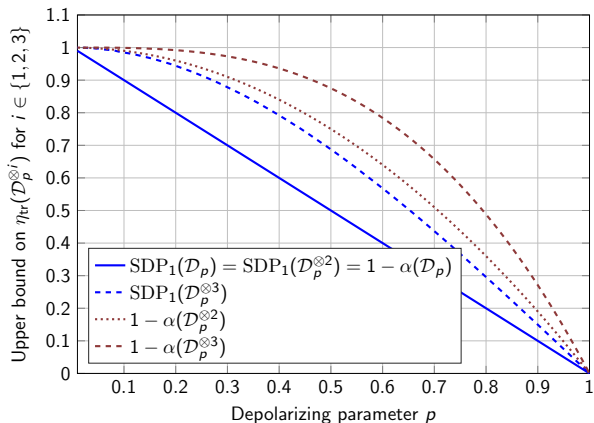
# SDP hierarchy converging to $\mathbb{P}_{\text{succ}}(\Phi, k)$ .

Theorem 3 proven using [Ber+21] for separability with linear constraints.

**Remark:**

- The size of our SDP hierarchy grows exponentially with the level.
- Quantum Doeblin Coefficient give efficient bounds (but not as tight) [Hir24; Geo+25]





**Figure:** Upper bounds on the contraction coefficient of (multiple copies of) depolarizing channels obtained via the first level of the hierarchy we propose.



# Conclusion

NP-**hardness** of:

- Approximate the trace norm contraction coefficient
- Find optimal encoding of a bit
- Decide if a quantum graph has independence number  $\geq 2$

**Future work:**

- Approximation algorithm based on separability results?
- Better algorithm if  $\Phi$  is structured?
- Encoding of a qubit (work in progress...)



# Thank you for your attention

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