

# Semantics and types for quantum Bayesian networks

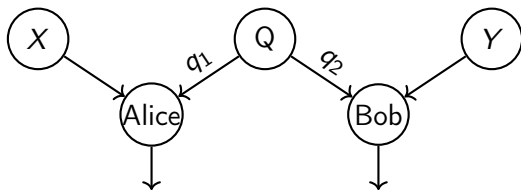
Rémi Di Guardia, Thomas Ehrhard, Claudia Faggian

IRIF (CNRS, Université Paris Cité)

Journées Informatique Quantique, 15 January 2026

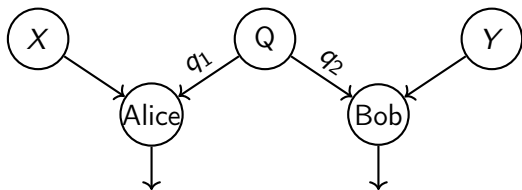


# Introduction – Bell's Experiment



Question: What is  $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$ ?

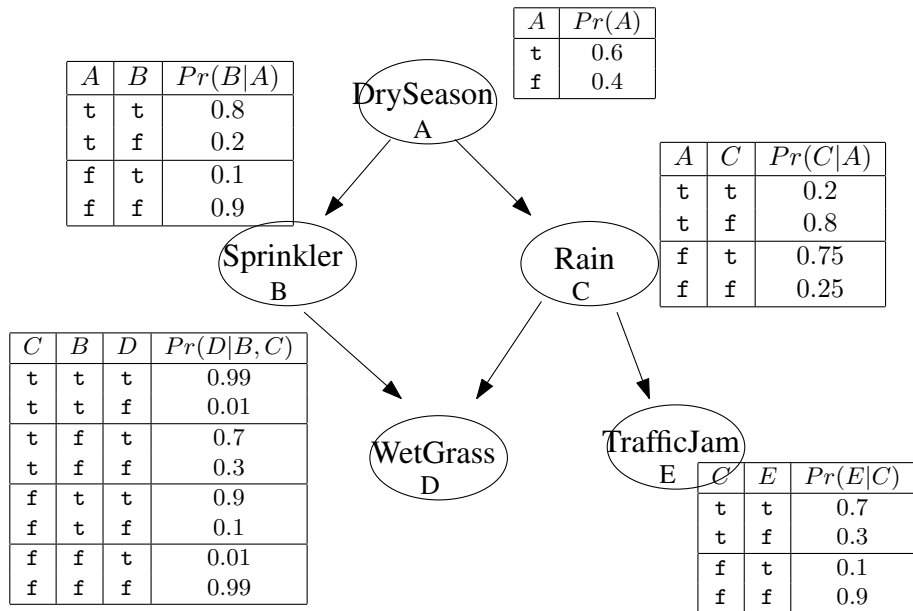
# Introduction – Bell's Experiment



Question: What is  $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$ ?

General Problem: Taking a quantum system, what is the probability of obtaining a given measurement?

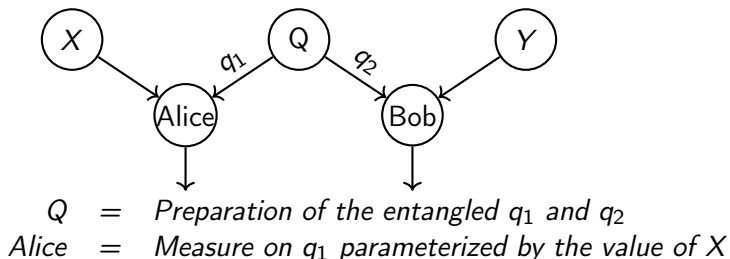
# Classical case – Bayesian Networks



# Quantum Bayesian Networks

## Definition: Quantum Bayesian Networks [HLP14]

DAG with a **quantum instrument** per node such that composing these instruments yields a probability distribution.



# Limits

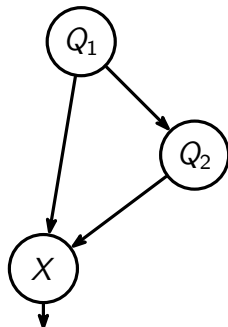
**Problems:** two wished results we do not have

- **Compositionality:** Given a decomposition of a network, is the data of the full network obtained from the data of each part?  
→ a big advantage of a graphical syntax, and a main property of Bayesian networks
- **Modularity:** Given two parts, can they compose to give a network?  
→ the result must be a DAG

*Our contributions:*

Solutions for these two problems, by giving another presentation of Quantum Bayesian Networks

# Compositionality Limit



$$\phi^{Q_1} : \mathbb{C} \rightarrow H_1 \otimes H_2$$

$$\phi^{Q_2} : H_2 \rightarrow H_3$$

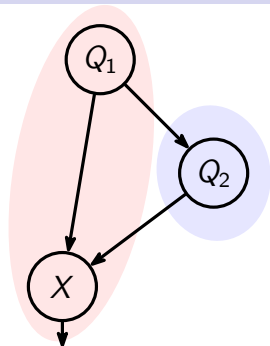
$$\phi^X(x) : H_1 \otimes H_3 \rightarrow \mathbb{C}$$

Whole graph:

$$\phi^{Q_1, Q_2, X}(x) : \mathbb{C} \rightarrow \mathbb{C}$$

$$= \phi^X(x) \circ (id_{H_1} \otimes \phi^{Q_2}) \circ \phi^{Q_1}$$

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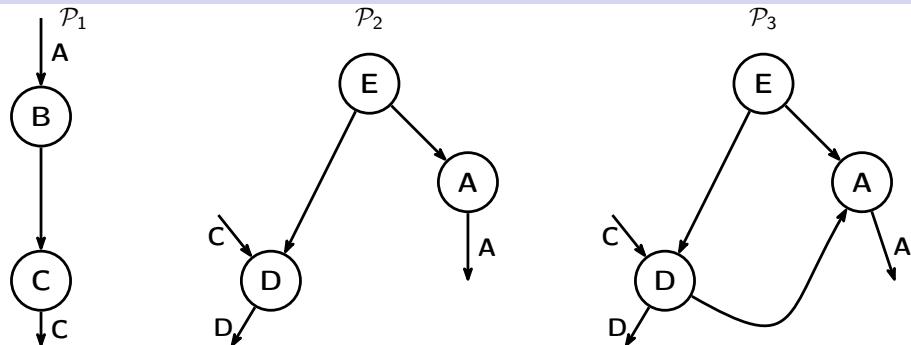
$$\phi^{Q_1, X}(x) : H_3 \rightarrow H_2 \quad \phi^{Q_2} : H_2 \rightarrow H_3$$

$\Rightarrow \phi^{Q_1, X, Q_2}(x)$  cannot simply be the **composition** of  $\phi^{Q_1, X}(x)$  and  $\phi^{Q_2}$ !  
(gives either  $H_3 \rightarrow H_3$  or  $H_2 \rightarrow H_2$ )

Idea: Functions do not work well at graph level, matrices would be better  
(also in Bayesian networks: factors instead of conditional probabilities)



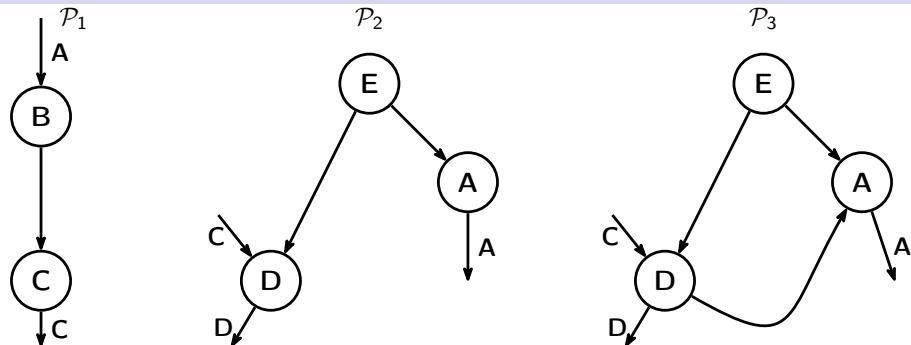
# Modularity Limit



- Part  $\mathcal{P}_1$  waits for input  $A$  and outputs  $C$
- Parts  $\mathcal{P}_2$  and  $\mathcal{P}_3$  wait for input  $C$  and output  $A$  and  $D$

Question: Is it “legal” to branch  $\mathcal{P}_1$  to  $\mathcal{P}_2$ ?  $\mathcal{P}_1$  to  $\mathcal{P}_3$ ?

# Modularity Limit



- Part  $\mathcal{P}_1$  waits for input A and outputs C
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Question: Is it “legal” to branch  $\mathcal{P}_1$  to  $\mathcal{P}_2$ ?  $\mathcal{P}_1$  to  $\mathcal{P}_3$ ?

→  $\mathcal{P}_1 \cup \mathcal{P}_2$  is a QBN but  $\mathcal{P}_1 \cup \mathcal{P}_3$  is *not* (cycle  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ )

Idea: Inputs & outputs are insufficient, we need a **type**

# Plan

- ▶ Compositionality by Quantum Factors
- ▶ Modularity by Typing

# Quantum Factors

Instead of associating a quantum instrument to a node associate a:

## Definition: Quantum Factor

Take random variables  $\mathbb{X} = (X_1, \dots, X_n)$ , and Hilbert spaces  $\mathbb{H} = (H_1, \dots, H_m)$ . A **Quantum Factor** on  $(\mathbb{X}, \mathbb{H})$  is a function  $\phi$  from  $Val(\mathbb{X}) = \prod_{i=1}^n Val(X_i)$  to *positive* matrices in  $\bigotimes_{j=1}^m H_j$ .

Equipped with a **product**  $\odot$ , such that for  $\phi_1$  and  $\phi_2$  respectively on  $(\mathbb{X}_1, \mathbb{H}_1)$  and  $(\mathbb{X}_2, \mathbb{H}_2)$ ,  $\phi_1 \odot \phi_2$  is on  $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{H}_1 \Delta \mathbb{H}_2)$ .

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Only classical: get factors from Bayesian networks and their product

# Quantum Factors

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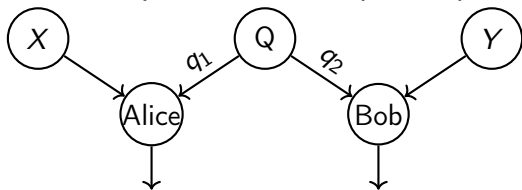
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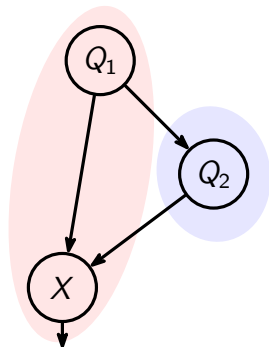
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quantum instrument  $\rightarrow$  quantum factor: pre-compose by cap



# Compositionality with quantum factors



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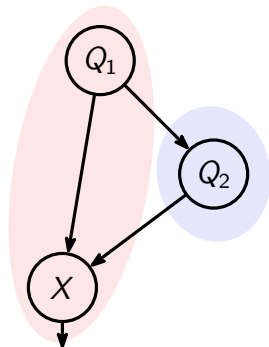
$$= \phi^X(x) \circ (id_{H_1} \otimes \phi^{Q_2}) \circ \phi^{Q_1}$$

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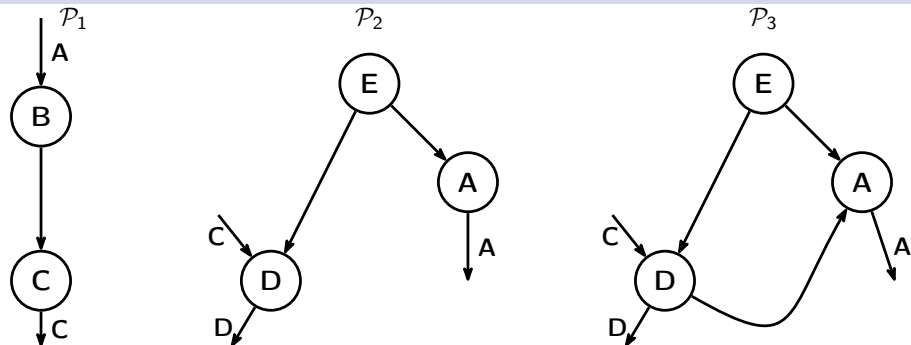
Whole graph:

$$\begin{aligned}\phi^{Q_1, Q_2, X} : \text{Val}(X) &\rightarrow \mathbb{C} \\ &= \phi^X \odot \phi^{Q_2} \odot \phi^{Q_1}\end{aligned}$$

$$\begin{aligned}\phi^{Q_1, X} : \text{Val}(X) &\rightarrow H_3 \otimes H_2 \\ \implies \phi^{Q_1, X, Q_2}(x) &= \phi^{Q_1, X} \odot \phi^{Q_2}\end{aligned}$$

More generally, can compute for any order on the nodes!

# Modularity



*Observation:*  $\mathcal{P}_1 \cup \mathcal{P}_2$  is a QBN but not  $\mathcal{P}_1 \cup \mathcal{P}_3$

*How to ensure two parts always form a QBN?*

We add a type = an interface

Could do so by patching QBN and getting yet another new syntax...

We prefer to use **proof-nets**, graphs from linear logic adapted to typing



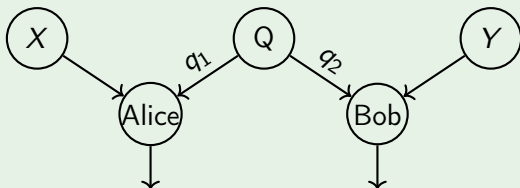
# Proof-Nets for Quantum Bayesian Networks

## Definition: Proof-Net

A graph respecting some *graphical criterion* and built from:

$$\begin{array}{c}
 A^\perp \left[ \begin{array}{c} ax \\ \hline \end{array} \right] A \quad A^\perp \left[ \begin{array}{c} cut \\ \hline \end{array} \right] A \quad X^- \left[ \begin{array}{c} c \\ \hline X^- \end{array} \right] X^- \quad \begin{array}{c} w \\ | \\ N \end{array} \quad \begin{array}{c} 1 \\ | \\ 1 \end{array} \quad \begin{array}{c} \perp \\ | \\ \perp \end{array} \\
 A \left[ \begin{array}{c} \otimes \\ \hline \end{array} \right] B \quad A \left[ \begin{array}{c} \wp \\ \hline \end{array} \right] B \quad \boxed{\phantom{...}} \quad \boxed{\phantom{...} Q} \\
 | A \otimes B \quad | A \wp B \quad Y_1^- \quad \dots \quad Q_1^- \quad X^+ \quad Y_1^- \quad \dots \quad Q_1^- \quad \otimes_i P_i^+
 \end{array}$$

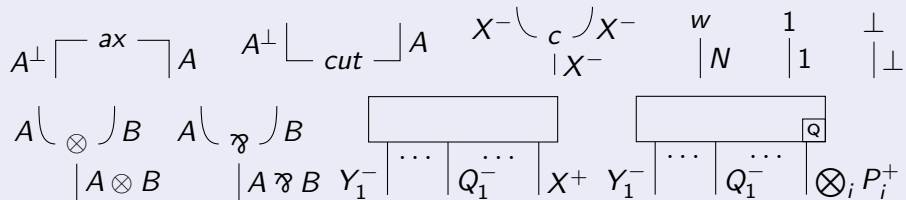
## Example



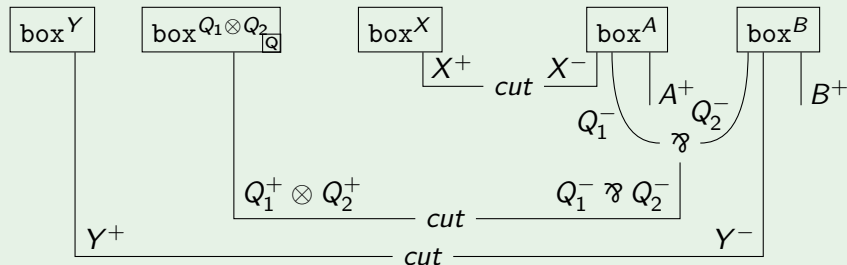
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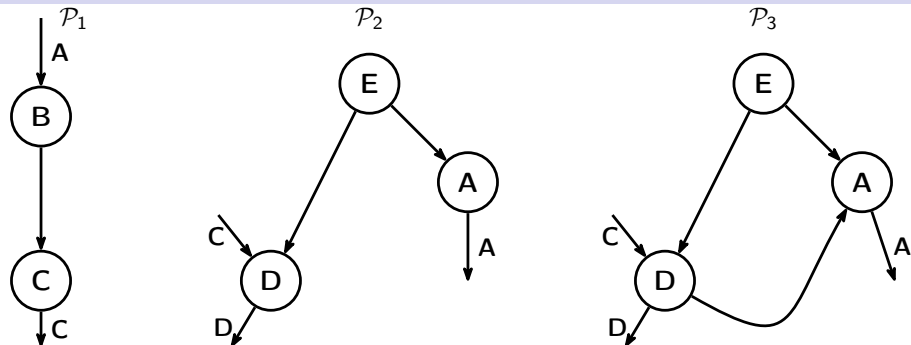
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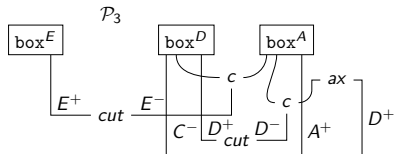
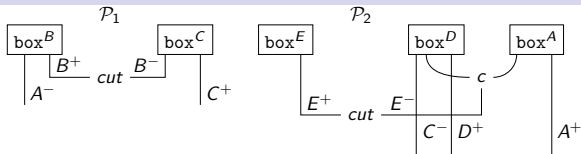


# Modularity with types



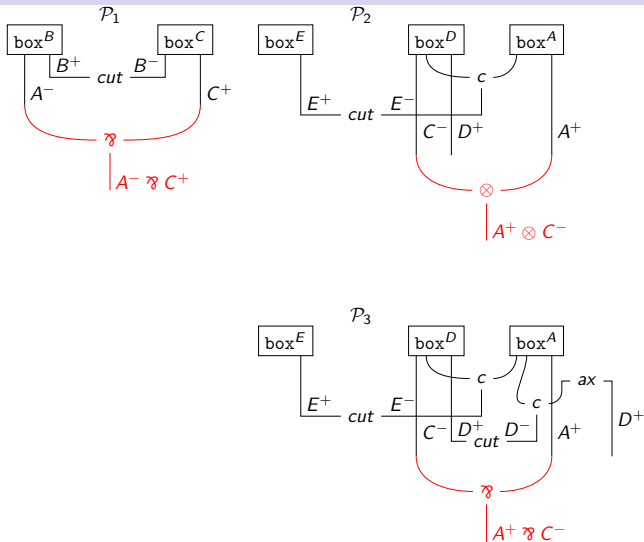
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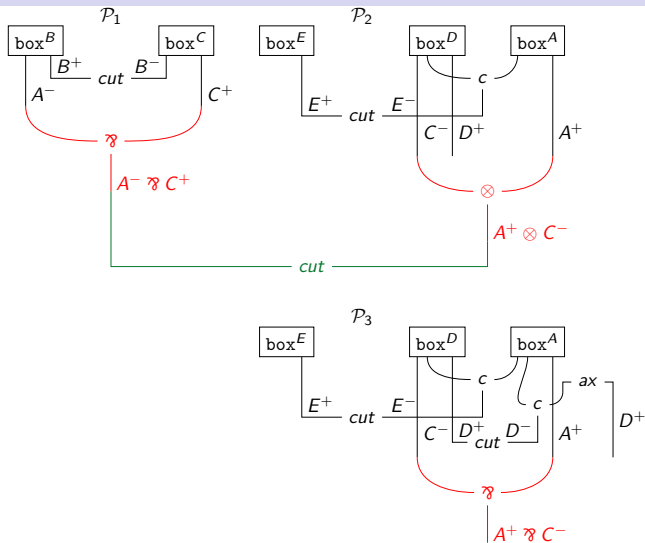
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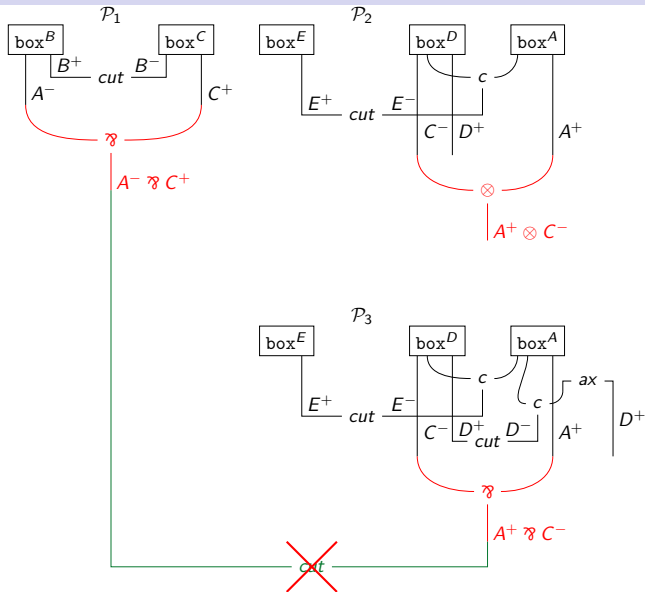
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# Conclusion

## Contributions

- **Compositionality** by modifying the semantic:  
from quantum instruments to (more general) *quantum factors*
- **Modularity** by typing in proof-nets:  
proof-theoretic approach adding an *interface*, parts with compatible interfaces are those giving a QBN



# Conclusion

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proof-theoretic approach adding an *interface*, parts with compatible interfaces are those giving a QBN

## Perspectives

- Compositionality can be used to study *conditional independence* (*no-signaling*) between random variables, even with quantum causes
- Modularity allows a weak form of *higher-order* (linear application), can we do more?
- Proof-Nets have *rewriting rules* (cut-elimination) corresponding to computations: can they be used to compute efficiently the quantum factor of the full network? (as in the classical case)

Thank you for  
your attention!

# References I

- [HLP14] Joe Henson, Raymond Lal, and Matthew F Pusey.  
“Theory-independent limits on correlations from generalized  
Bayesian networks”. In: *New Journal of Physics* 16.11 (Nov.  
2014). ISSN: 1367-2630. DOI:  
10.1088/1367-2630/16/11/113043.