

Semantics and types for quantum Bayesian networks

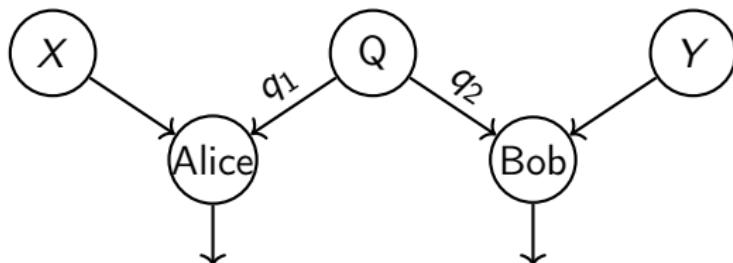
Rémi Di Guardia, Thomas Ehrhard, Claudia Faggian

IRIF (CNRS, Université Paris Cité)

Journées Informatique Quantique, 15 January 2026

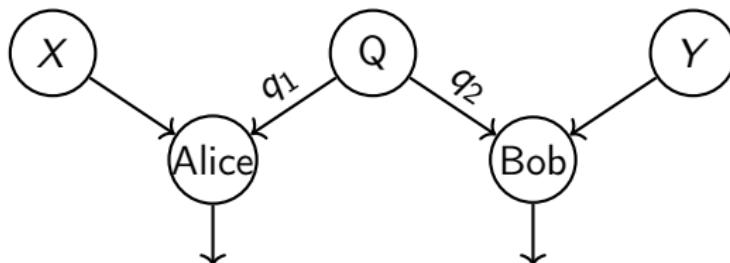


Introduction – Bell's Experiment



Question: What is $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

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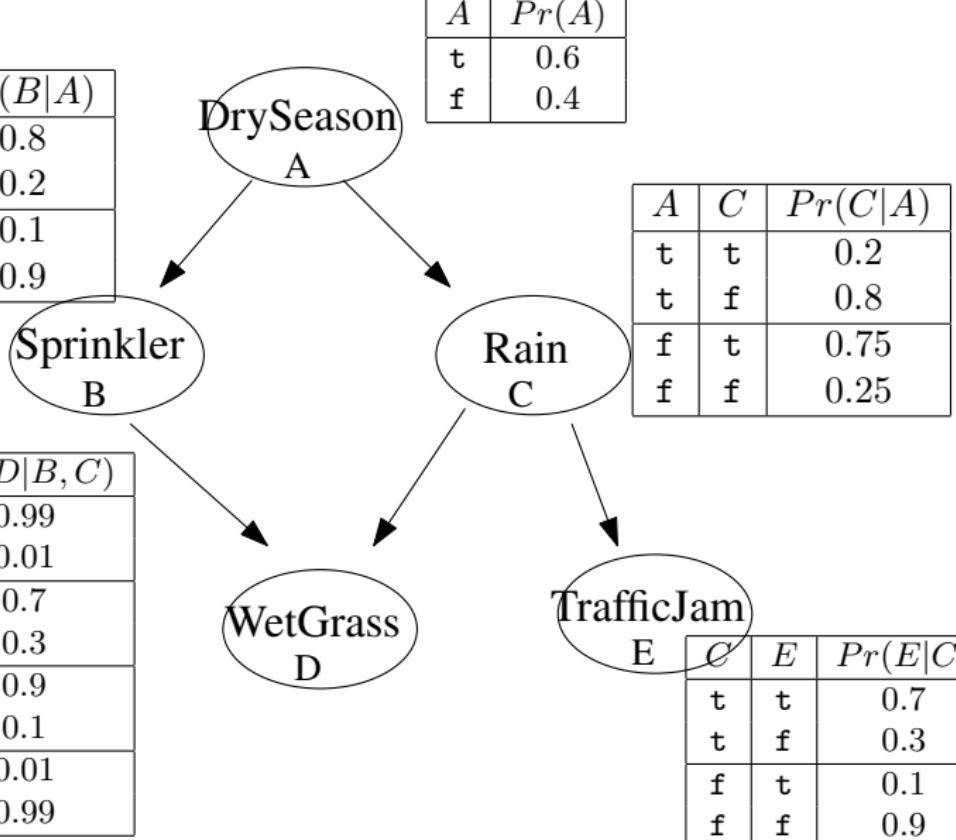


Question: What is $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

General Problem: Taking a quantum system, what is the probability of obtaining a given measurement?

Classical case – Bayesian Networks

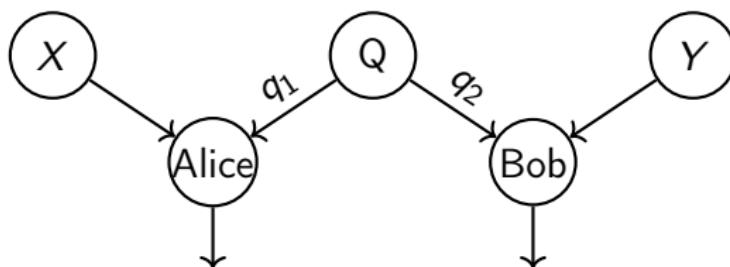
A	B	$Pr(B A)$
t	t	0.8
t	f	0.2
f	t	0.1
f	f	0.9



Quantum Bayesian Networks

Definition: Quantum Bayesian Networks [HLP14]

DAG with a **quantum instrument** per node such that composing these instruments yields a probability distribution.



Q = Preparation of the entangled q_1 and q_2

$Alice$ = Measure on q_1 parameterized by the value of X

Limits

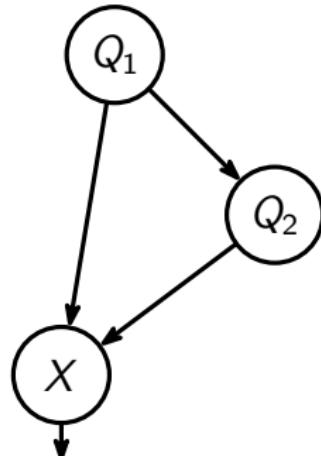
Problems: two wished results we do not have

- **Compositionality:** Given a decomposition of a network, is the data of the full network obtained from the data of each part?
→ a big advantage of a graphical syntax, and a main property of Bayesian networks
- **Modularity:** Given two parts, can they compose to give a network?
→ the result must be a DAG

Our contributions:

Solutions for these two problems, by giving another presentation of Quantum Bayesian Networks

Compositionality Limit



$$\phi^{Q_1} : \mathbb{C} \rightarrow H_1 \otimes H_2$$

$$\phi^{Q_2} : H_2 \rightarrow H_3$$

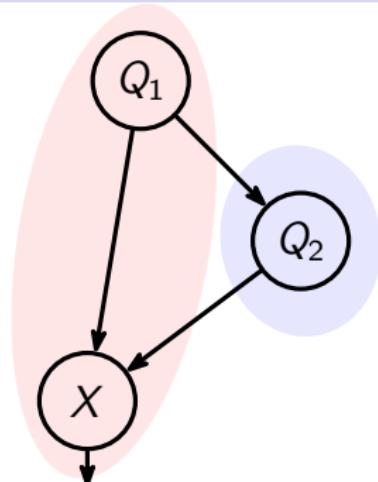
$$\phi^X(x) : H_1 \otimes H_3 \rightarrow \mathbb{C}$$

Whole graph:

$$\phi^{Q_1, Q_2, X}(x) : \mathbb{C} \rightarrow \mathbb{C}$$

$$= \phi^X(x) \circ (id_{H_1} \otimes \phi^{Q_2}) \circ \phi^{Q_1}$$

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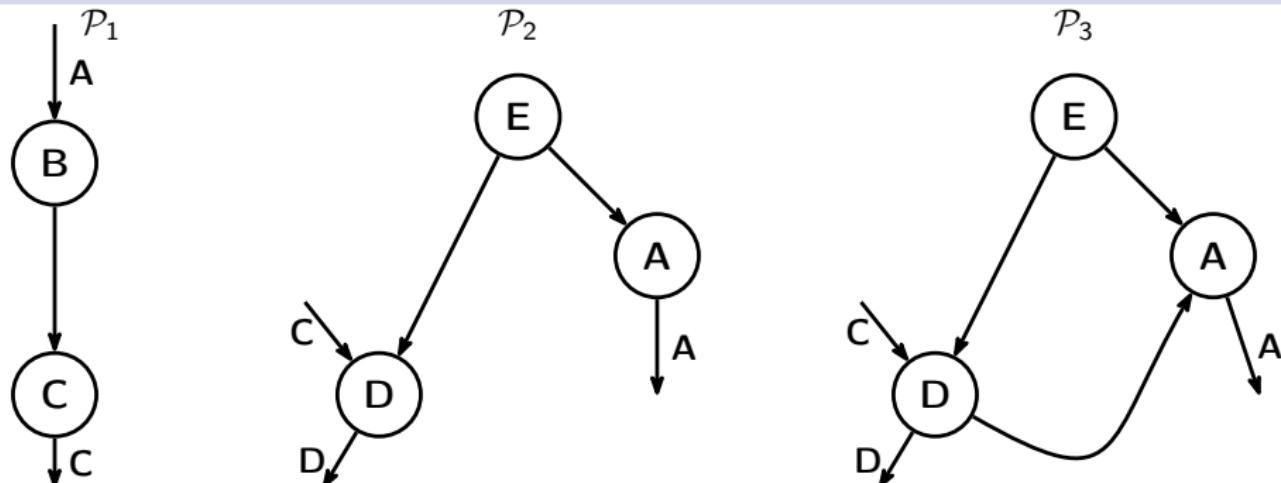
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$$\phi^{Q_1, X}(x) : H_3 \rightarrow H_2 \quad \phi^{Q_2} : H_2 \rightarrow H_3$$

$\implies \phi^{Q_1, X, Q_2}(x)$ cannot simply be the **composition** of $\phi^{Q_1, X}(x)$ and ϕ^{Q_2} !
(gives either $H_3 \rightarrow H_3$ or $H_2 \rightarrow H_2$)

Idea: Functions do not work well at graph level, matrices would be better
(also in Bayesian networks: factors instead of conditional probabilities)

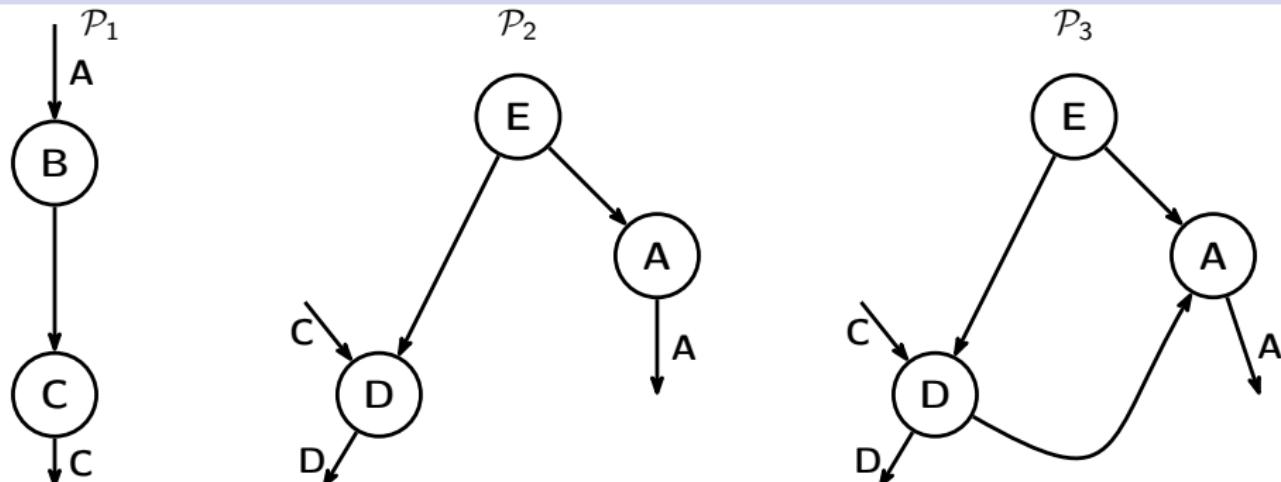
Modularity Limit



- Part \mathcal{P}_1 waits for input A and outputs C
- Parts \mathcal{P}_2 and \mathcal{P}_3 wait for input C and output A and D

Question: Is it “legal” to branch \mathcal{P}_1 to \mathcal{P}_2 ? \mathcal{P}_1 to \mathcal{P}_3 ?

Modularity Limit



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- Parts \mathcal{P}_2 and \mathcal{P}_3 wait for input C and output A and D

Question: Is it “legal” to branch \mathcal{P}_1 to \mathcal{P}_2 ? \mathcal{P}_1 to \mathcal{P}_3 ?

→ $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but $\mathcal{P}_1 \cup \mathcal{P}_3$ is *not* (cycle $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$)

Idea: Inputs & outputs are insufficient, we need a **type**

Plan

- ▶ Compositionality by Quantum Factors
- ▶ Modularity by Typing

Quantum Factors

Instead of associating a quantum instrument to a node associate a:

Definition: Quantum Factor

Take random variables $\mathbb{X} = (X_1, \dots, X_n)$, and Hilbert spaces $\mathbb{H} = (H_1, \dots, H_m)$. A **Quantum Factor** on (\mathbb{X}, \mathbb{H}) is a function ϕ from $Val(\mathbb{X}) = \prod_{i=1}^n Val(X_i)$ to *positive* matrices in $\bigotimes_{j=1}^m H_j$.

Equipped with a **product** \odot , such that for ϕ_1 and ϕ_2 respectively on $(\mathbb{X}_1, \mathbb{H}_1)$ and $(\mathbb{X}_2, \mathbb{H}_2)$, $\phi_1 \odot \phi_2$ is on $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{H}_1 \Delta \mathbb{H}_2)$.

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Only classical: get factors from Bayesian networks and their product

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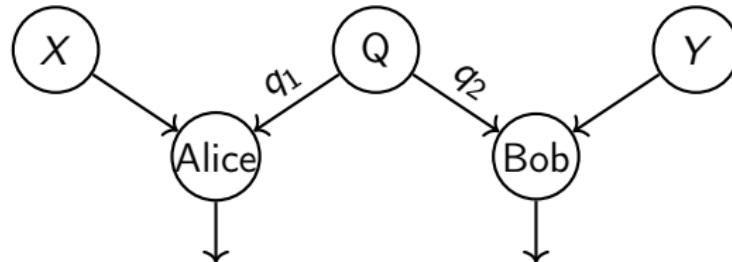
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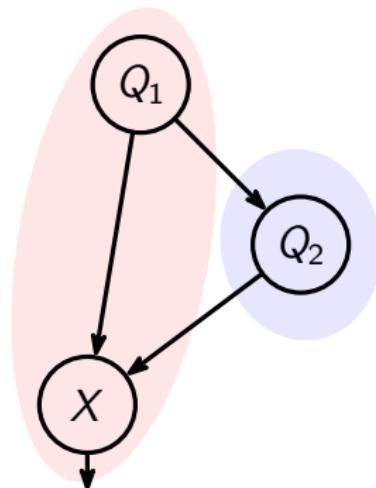
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quantum instrument \rightarrow **quantum factor**: pre-compose by cap



Compositionality with quantum factors



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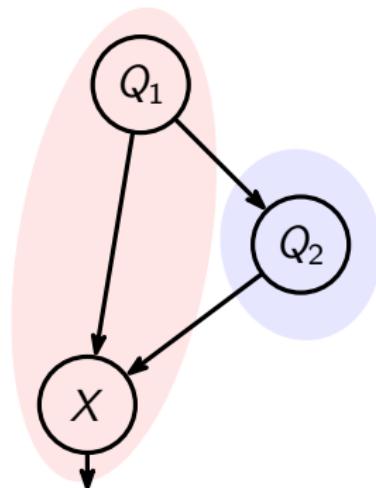
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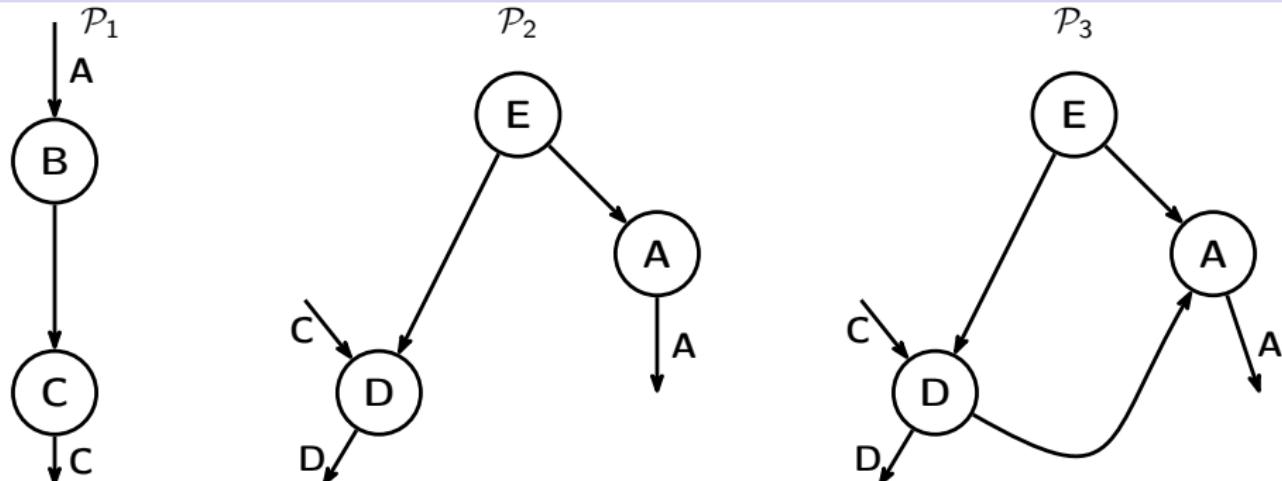
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$$\implies \phi^{Q_1, X, Q_2}(x) = \phi^{Q_1, X} \odot \phi^{Q_2}$$

More generally, can compute for any order on the nodes!

Modularity



Observation: $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but not $\mathcal{P}_1 \cup \mathcal{P}_3$

How to ensure two parts always form a QBN?

We add a type = an interface

Could do so by patching QBN and getting yet another new syntax...

We prefer to use **proof-nets**, graphs from linear logic adapted to typing

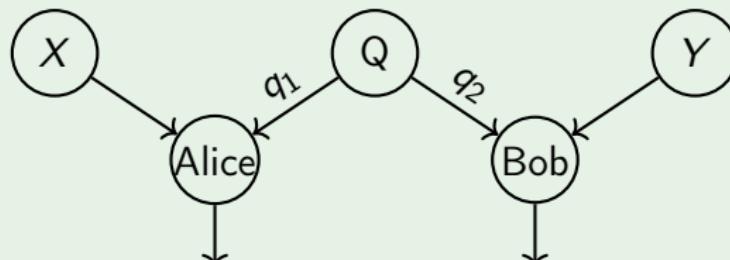
Proof-Nets for Quantum Bayesian Networks

Definition: Proof-Net

A graph respecting some *graphical criterion* and built from:

$$\begin{array}{ccccccccc} A^\perp \lceil \text{ } ax \text{ } \rceil A & A^\perp \lceil \text{ } cut \text{ } \rceil A & X^- \backslash \text{ } c \text{ } / X^- & w & 1 & \perp & & & \\ \text{ } & \text{ } & | & | & | & | & & & \\ A \backslash \text{ } \otimes \text{ } / B & A \backslash \text{ } \wp \text{ } / B & \text{ } & \\ | & | & \dots & \dots & \dots & \dots & \dots & \dots & \\ A \otimes B & A \wp B & Y_1^- & Q_1^- & X^+ & Y_1^- & Q_1^- & \otimes_i P_i^+ & \end{array}$$

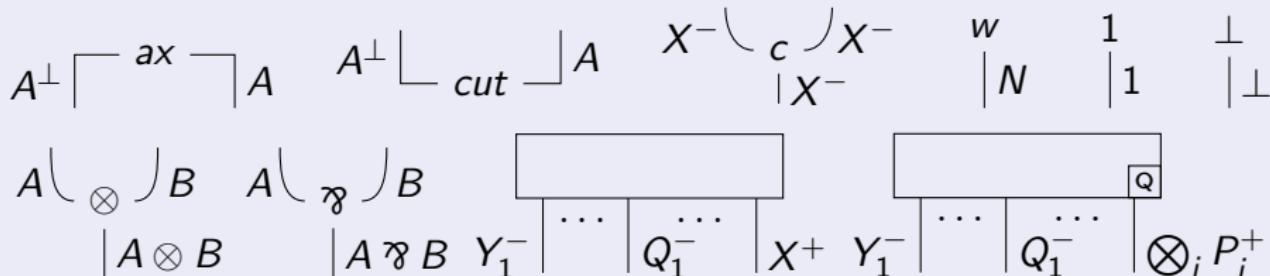
Example



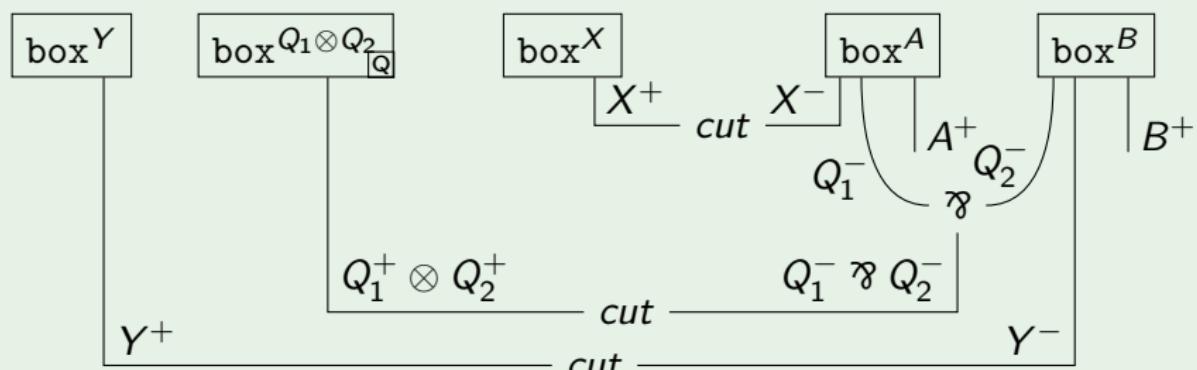
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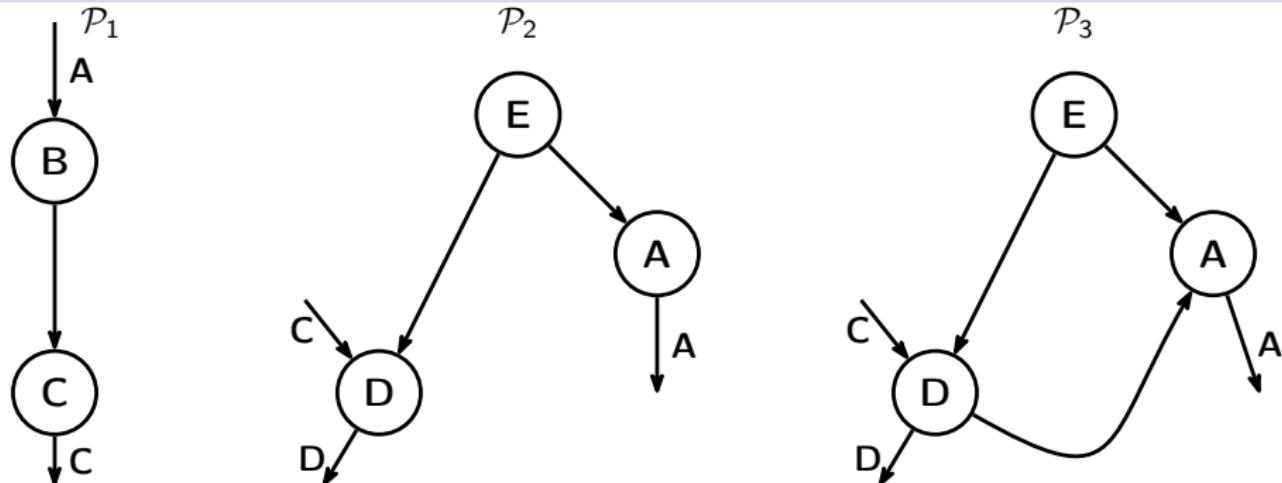
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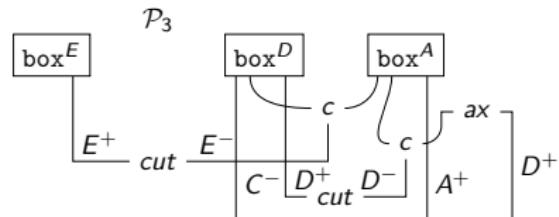
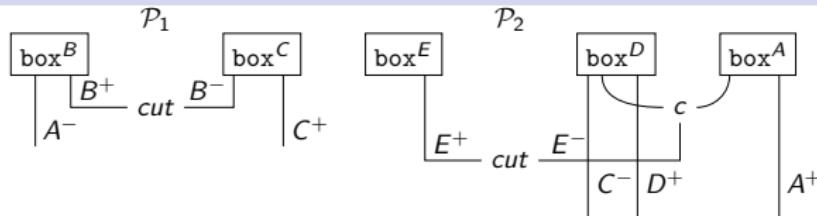


Modularity with types



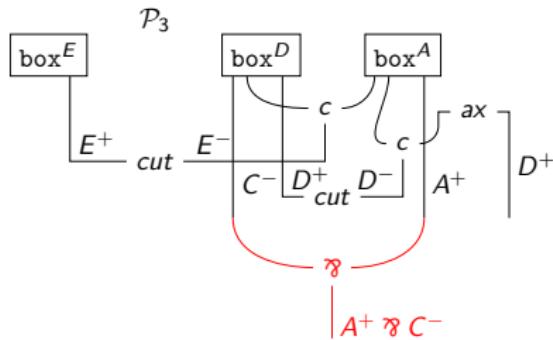
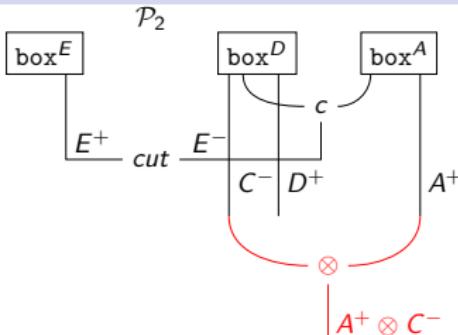
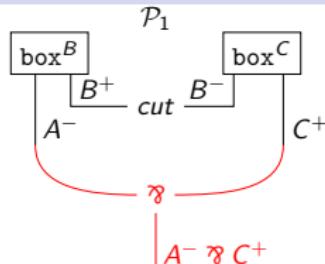
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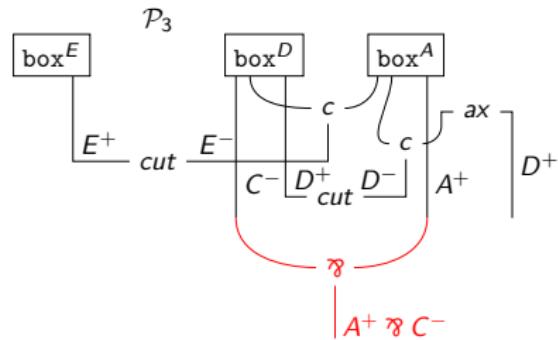
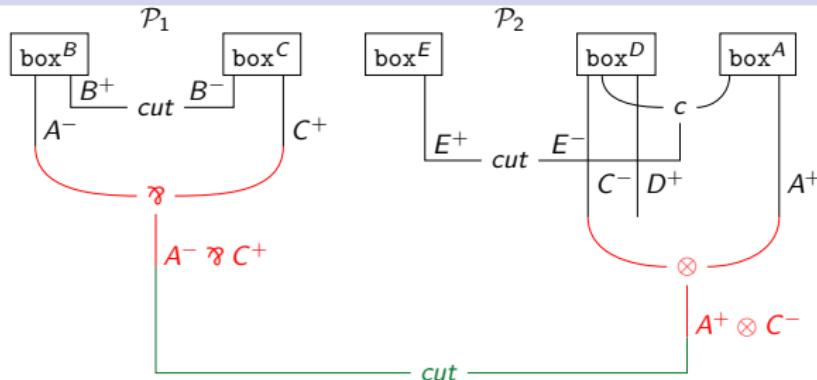
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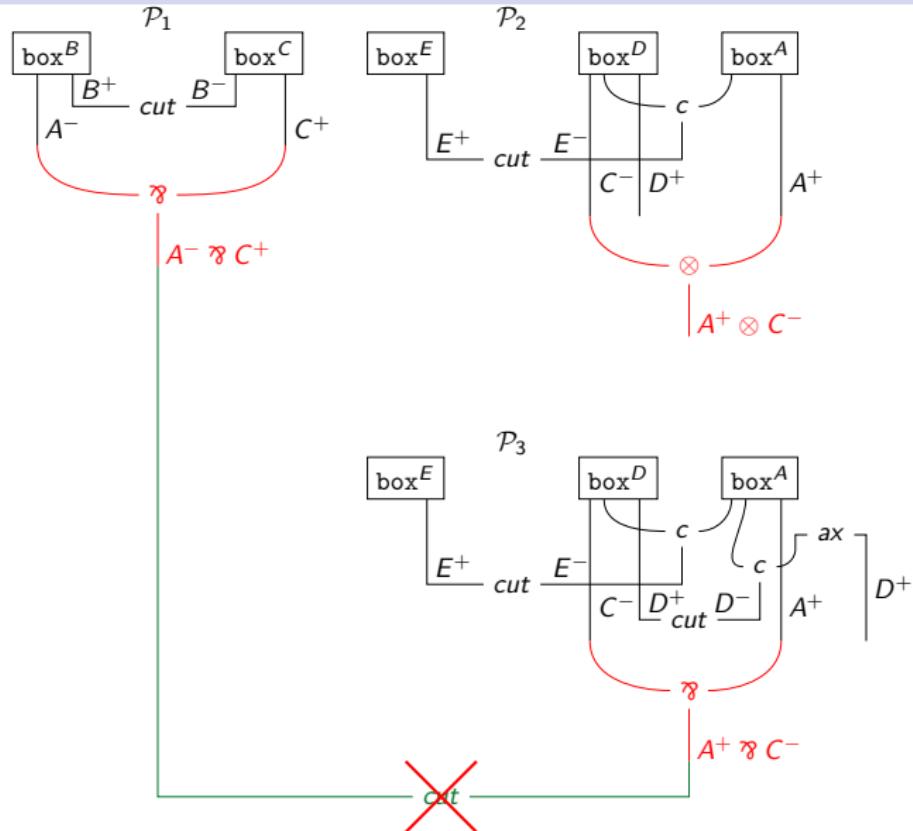
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Conclusion

Contributions

- **Compositionality** by modifying the semantic:
from quantum instruments to (more general) *quantum factors*
- **Modularity** by typing in proof-nets:
proof-theoretic approach adding an *interface*, parts with compatible
interfaces are those giving a QBN

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Perspectives

- Compositionality can be used to study *conditional independence* (*no-signaling*) between random variables, even with quantum causes
- Modularity allows a weak form of *higher-order* (linear application), can we do more?
- Proof-Nets have *rewriting rules* (cut-elimination) corresponding to computations: can they be used to compute efficiently the quantum factor of the full network? (as in the classical case)

Thank you for
your attention!

References I

[HLP14] Joe Henson, Raymond Lal, and Matthew F Pusey.
"Theory-independent limits on correlations from generalized
Bayesian networks". In: *New Journal of Physics* 16.11 (Nov.
2014). ISSN: 1367-2630. DOI:
[10.1088/1367-2630/16/11/113043](https://doi.org/10.1088/1367-2630/16/11/113043).